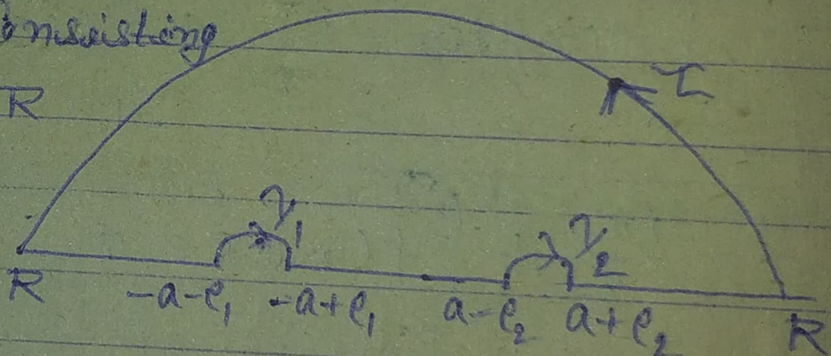
M.U. M.Sc. 96

Q.No → Prove that if $a>0$, $\int_{-\infty}^{\infty} \frac{\cos x}{a^2-x^2}\,dx = \frac{\pi}{a}\sin a$

Soln:- We consider the integral $\int_C \frac{e^{iz}}{a^2-z^2} = \int_C f(z)\,dz$

where C is contour

~~Where C is~~ Consisting of real axis from $-R$ to $R_1 - \ell_1$; the semi circle γ_1 of small radius ℓ_1 the real axis $-a+\ell_1$ to $a-\ell_2$, the semi circle γ_2 of small radius ℓ_2, the real axis from $a+\ell_2$ to R together with the semi circle Γ of large radius R, Hence,



$$\int_C f(z)\,dz = \int_{-R}^{-a-\ell_1} f(x)\,dx + \int_{\gamma_1} f(z)\,dz + \int_{-a+\ell_1}^{a-\ell_2} f(x)\,dx$$
$$+ \int_{\gamma_2} f(z)\,dz + \int_{a+\ell_2}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

As $f(z)$ has no singularities within C.

Since, $\underset{R\to 0}{Lt} \int_{\Gamma} \frac{e^{iz}}{a^2-z^2}\,dz$

$$= \underset{R\to\infty}{Lt} \int_{\Gamma} f(z)\,dz = 0$$

$$= \underset{z\to -a}{Lt} (z+a) f(z) = \underset{z\to -a}{Lt} \frac{(z+a)e^{iz}}{a^2-z^2} = \underset{z\to -a}{Lt} \frac{e^{iz}}{a-z}$$

$$= \frac{e^{ia}}{2a}$$

$$\therefore \underset{\rho_1\to 0}{Lt} \int_{\gamma_1} f(z)\,dz = i(0-\pi)\cdot\frac{e^{-ia}}{2a} = -\frac{i\pi e^{-ia}}{2a}$$

Also, $\underset{z\to a}{Lt} (z-a) f(z) = \underset{z\to -a}{Lt} \frac{(z-a)e^{iz}}{a^2-z^2}$

$$= \underset{z\to a}{Lt} \frac{-e^{iz}}{a+z} = -\frac{e^{ia}}{2a}$$

$$\therefore \underset{\rho_2\to 0}{Lt} \int_{\gamma_2} f(z)\,dz = -i(0-\pi)\frac{e^{ia}}{2a} = -\frac{i\pi e^{ia}}{2a}$$

$$\therefore \int_{-\infty}^{-a} f(x)\,dx + \frac{-i\pi e^{-ia}}{2a} + \int_{-a}^{a} f(x)\,dx + \frac{i\pi e^{ia}}{2a}$$

$$+ \int_{0}^{\infty} f(x)\,dx = 0$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = -\frac{i\pi}{2a}\left(e^{ia} - e^{-ia}\right)$$

$$= -\frac{i\pi}{2a} \cdot 2i\sin a = \frac{2\pi\sin a}{2a}$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{ix}}{a^2-x^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{a-x^2} = \frac{\pi\sin a}{a}$$

Equating the real Part from both Sides, we have

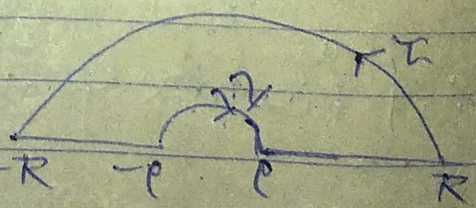
$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2-x^2}\,dx = \frac{\pi\sin a}{a} \quad \text{Proved.}$$

QNo→ Use the method of Contour integration to Prove that,

$$\int_0^{\infty} \frac{x^b}{1+x^2}\,dx = \frac{\pi}{2}\sec\frac{\pi b}{2} \quad (-1<b<1).$$

Soln:- We Consider the integral,

$$\int_C \frac{z^b}{1+z^2}\,dz = \int_C f(z)\,dz$$ where C is Contour



Consisting of the real axis from $-R$ to $-\rho$ & semi Circle γ of Small radius ρ, real axis from ρ to R and the Semi Circle Γ of large radius R

$$\therefore \int_C f(z)\,dz = \int_{-R}^{-\rho} f(x)\,dx + \int_{\gamma} f(z)\,dz + \int_{\rho}^{R} f(x)\,dx$$

$+\int_{\Gamma} f(z)\,dz = 2\pi i \sum R^+$. Since, $-1 < b < 1$.

$$\underset{z\to\infty}{Lt}\; z f(z) = \underset{z\to\infty}{Lt}\; \frac{z \cdot z^b}{1+z^2} = 0$$

$$\underset{R\to\infty}{Lt} \int_{\Gamma} f(z)\,dz = 0 \text{ and } \underset{z\to 0}{Lt}\; z f(z) = \underset{z\to 0}{Lt}\; \frac{z \cdot z^b}{1+z^2} = 0$$

$$\therefore \underset{\rho\to 0}{Lt} \int_{\gamma} f(z)\,dz = 0$$

$$\therefore \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 2\pi i \sum R^+.$$

The Poles of $f(z)$ are given by $1+z^2=0$
$\therefore z = \pm i$ of which only $z=i$ lies within C. the residue $z=i$

$$= \underset{z\to i}{Lt}\; (z-i) f(z) = \underset{z\to i}{Lt}\; (z-i) \cdot \frac{z^b}{1+z^2}$$

$$= \underset{z\to i}{Lt}\; (z-i) \frac{z^b}{(z+i)(z-i)}$$

$$= \underset{z\to i}{Lt}\; \frac{z^b}{z+i} = \frac{1}{2i} i^b = \frac{1}{2i} e^{b\log i}$$

$$= \frac{1}{2i} e^{b\log e^{i\pi/2}} = \frac{1}{2i} e^{ib\pi/2}$$

$$\text{Therefore, } \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 2\pi i \sum R^+$$

$$= 2\pi i \cdot \frac{e^{\frac{i\pi b}{2}}}{2i} = \pi e^{i\pi b/2}$$

Since, on the -ve Part of real axes x will be -ve, ∴ So may take $x = y.e^{i\pi}$

∴ $dx = e^{i\pi} dy$ and the limit will be ∞ to 0.

$$\therefore \int_{-\infty}^{0} f(x)\,dx = \int_{-\infty}^{0} \frac{x^b\,dx}{1+x^2}$$

$$= \int_{\infty}^{0} \frac{e^{i\pi b}\, y^b \cdot e^{i\pi} dy}{1+y^2 e^{2i\pi}} = \int_{0}^{\infty} \frac{e^{i\pi b} . x^b\, dx}{1+x^2}$$

$$\therefore \int_{0}^{\infty} \frac{e^{i\pi b} . x^b}{1+x^2}\,dx + \int_{0}^{\infty} \frac{x^b}{1+x^2}\,dx = \pi . e^{i\pi . \frac{b}{2}}$$

$$\therefore \int_{0}^{\infty} \frac{x^b}{1+x^2}(1+\cos\pi b + i\sin\pi b)\,dx$$

$$= \pi\left(\cos\frac{\pi b}{2} + i\sin\frac{\pi b}{2}\right)$$

Equating real Part from both Sides, we have

$$\int_{0}^{\infty} (1+\cos\pi b)\frac{x^b}{1+x^2}\,dx = \pi\cos\frac{\pi b}{2}$$

$$\therefore \int_{0}^{\infty} \frac{x^b}{1+x^2}\,dx = \frac{\pi\cos\pi\frac{b}{2}}{1+\cos\pi b} = \frac{\pi\cos\pi\frac{b}{2}}{2\cos^2\frac{\pi b}{2}}$$

$$= \frac{\pi}{2}\sec\frac{\pi b}{2}$$, Proved.

QNo → Prove by Contour integration,

$$\int_0^1 \frac{\log\left(x+\frac{1}{x}\right)}{1+x^2}\,dx = \frac{\pi}{2}\log 2.$$

Solⁿ :- $I = \int_0^1 \frac{\log\left(x+\frac{1}{x}\right)}{1+x^2}\,dx = \int_0^\infty \frac{\log\left(x+\frac{1}{x}\right)}{1+x^2}\,dx$

$$+\int_\infty^1 \frac{\log\left(x+\frac{1}{x}\right)}{1+x^2}\,dx$$

In the 2'nd integral on the R.H.S. be put $x = \frac{1}{t}$ ∴ $dx = -\frac{dt}{t^2}$ and limit will be 0 to 1.

$$\therefore \int_0^1 \frac{\log\left(x+\frac{1}{x}\right)}{1+x^2}\,dx = \int_0^1 \frac{\log\left(\frac{1}{t}+t\right)dt}{t^2\left(1+\frac{1}{t^2}\right)}$$

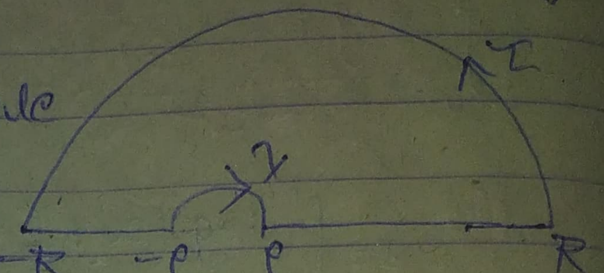
$$= -\int_0^1 \frac{\log\left(x+\frac{1}{x}\right)dx}{1+x^2} = -I$$

$$\therefore I = \int_0^\infty \frac{\log\left(x+\frac{1}{x}\right)}{1+x^2}\,dx - I \quad \therefore 2I = \int_0^\infty \frac{\log\frac{x^2+1}{x}}{x^2+1}\,dx$$

$$= \int_0^\infty \frac{\log(x^2+1)}{x^2+1}\,dx - \int_0^\infty \frac{\log x}{x^2+1}\,dx$$

We Consider $\int_c \frac{\log z}{z^2+1}\,dz = \int_c f(z)\,dz$

Where C is the Contour Consisting of the real axis $-R$ to $-\rho$, the semi circle γ of small radius ρ, the real axis from ρ to R and the semi circle Γ of large radius R.



$$\therefore \int_C f(z)\,dz = \int_{-R}^{-\rho} f(x)\,dx + \int_{\gamma} f(z)\,dz + \int_{\rho}^{R} f(x)\,dx$$

$$+ \int_{\Gamma} f(z)\,dz = 2\pi i \sum R^+$$

$$\because \lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z \cdot \log z}{z^2+1} = 0$$

and $$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{z\log z}{1+z^2} = 0$$

We have

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0 \quad \lim_{\rho\to 0} \int_{\gamma} f(z)\,dz = 0$$

$$\therefore \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 2\pi i \sum R^+$$

The Poles of $f(z)$ are given by $z^2+1=0$ i.e. $z=\pm i$ of which only $z=i$ lies within C.

$$\therefore \sum R^+ = \lim_{z\to i} (z-i) f(z) = \lim_{z\to i} (z-i) \frac{\log z}{z^2+1}$$

$$= \lim_{z \to i} \frac{(z-i)\log z}{(z-i)(z+i)} = \frac{\log i}{2i}$$

$$= \frac{\log e^{i\pi/2}}{2i} = \frac{i\pi}{2 \times 2i} = \frac{\pi}{4}$$

$$\therefore \int_{-\infty}^{0} \frac{\log x}{1+x^2}\,dx + \int_{0}^{\infty} \frac{\log x}{1+x^2}\,dx = 2\pi i \frac{\pi}{4}.$$

We know that on the $-ve$ Part of real axis x can be replace by $xe^{i\pi}$.

$$\therefore \int_{0}^{\infty} \frac{\log x}{1+x^2} + \int_{\infty}^{0} \frac{\log x e^{i\pi}}{1+x^2 e^{2i\pi}} e^{i\pi}\,dx = \frac{\pi^2 i}{2}.$$

$$\therefore \int_{0}^{\infty} \frac{\log x}{1+x^2}\,dx + \int_{0}^{\infty} \frac{\log x}{1+x^2}\,dx + \int_{0}^{\infty} \frac{\log e^{i\pi}}{1+x^2}\,dx = \frac{i\pi^2}{2}.$$

Equating the real Part, we have

$$2\int_{0}^{\infty} \frac{\log x}{1+x^2}\,dx = 0 \quad i.e. \int_{0}^{\infty} \frac{\log x}{1+x^2}\,dx = 0$$

$$\text{Hence, } 2I = \int_{0}^{\infty} \frac{\log(1+x^2)}{1+x^2}\,dx = \pi \log 2;$$

$$\therefore I = \int_{0}^{\infty} \frac{\log\left(x + \frac{1}{x}\right)}{1+x^2}\,dx = \frac{\pi}{2}\log 2.$$